

Multi-parameter projection theorems with applications to sums-products and finite point configurations in the Euclidean setting

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June 29, 2011

Abstract

In this paper we study multi-parameter projection theorems for fractal sets. With the help of these estimates, we recover results about the size of $A \cdot A + \cdots + A \cdot A$, where A is a subset of the real line of a given Hausdorff dimension, $A + A = \{a + a' : a, a' \in A\}$ and $A \cdot A = \{a \cdot a' : a, a' \in A\}$. We also use projection results and inductive arguments to show that if a Hausdorff dimension of a subset of \mathbb{R}^d is sufficiently large, then the $\binom{k+1}{2}$ -dimensional Lebesgue measure of the set of k -simplexes determined by this set is positive. The sharpness of these results and connection with number theoretic estimates is also discussed.

1 Introduction

We start out by briefly reviewing the underpinnings of the sum-product results in the discrete setting. A classical conjecture in geometric combinatorics is that either the sum-set or the product-set of a finite subset of the integers is maximally large. More precisely, let $A \subset \mathbb{Z}$ of size N and define

$$A + A = \{a + a' : a, a' \in A\}; \quad A \cdot A = \{a \cdot a' : a, a' \in A\}.$$

The Erdos-Szemerédi conjecture says that

$$\max\{\#(A + A), \#(A \cdot A)\} \gtrsim N^2,$$

where here, and throughout, $X \lesssim Y$ means that there exists $C > 0$ such that $X \leq CY$ and $X \gtrsim Y$, with the controlling parameter N if for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon N^\epsilon Y$. The best currently known result is due to Jozsef Solymosi ([19]) who proved that

$$\max\{\#(A + A), \#(A \cdot A)\} \gtrsim N^{\frac{4}{3}}. \tag{1.1}$$

For the finite field analogs of these problems, see, for example, [2], [8], [9], [12], [10], [11] and [20].

The sum product problem has also received considerable amount of attention in the Euclidean setting. The following result was proved by Edgar and Miller ([4]) and, independently, by Bourgain ([1]).

Theorem 1.0.1. *A Borel sub-ring of the reals either has Hausdorff dimension 0 or is all of the real line.*

Bourgain ([1]) proved the following quantitative bound that was conjectured in ([14]).

Theorem 1.0.2. *Suppose that $A \subset \mathbb{R}$ is a (δ, σ) -set in the sense that A is a union of δ -intervals and for $0 < \epsilon \ll 1$,*

$$|A \cap I| < \left(\frac{r}{\delta}\right)^{1-\sigma} \delta^{1-\epsilon}$$

whenever I is an arbitrary interval of size $\delta \leq r \leq 1$. Suppose that $0 < \sigma < 1$ and $|A| > \delta^{\sigma+\epsilon}$. Then

$$|A + A| + |A \cdot A| > \delta^{\sigma-c}$$

for an absolute constant $c = c(\sigma) > 0$.

One of the key steps in the proof is the study of the size of

$$A \cdot A - A \cdot A + A \cdot A - A \cdot A$$

and this brings us to the main results of this paper. Our goal is to show that if the Hausdorff dimension of $A \subset \mathbb{R}$ is sufficiently large, then

$$\mathcal{L}^1(a_1 A + a_2 A + \cdots + a_d A) > 0$$

for a generic choice of $(a_1, \dots, a_d) \in A \times A \times \cdots \times A$. It is of note that much work has been done in this direction in the setting of finite fields. See [?], [10] and the references contained therein. In particular, it is the work in [?] that inspired some of the following results.

Our results are proved using generalized projections theorems, similar in flavor to the ones previously obtained by Peres and Schlag ([17]) and Solomyak ([18]).

Theorem 1.0.3. *Let $E, F \subset \mathbb{R}^d$, $d \geq 2$, be of Hausdorff dimension s_E, s_F , respectively. Suppose that there exist Frostman measures μ_E, μ_F , supported on E and F , respectively, such that for δ sufficiently small and $|\xi|$ sufficiently large, there exist non-negative numbers γ_F and l_F such that the following conditions hold:*

•

$$|\widehat{\mu}_F(\xi)| \lesssim |\xi|^{-\gamma_F} \tag{1.2}$$

and

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$$\mu_F(T_\delta) \lesssim \delta^{s_F - l_F}, \quad (1.3)$$

for every tube T_δ of length ≈ 1 and radius $\approx \delta$ emanating from the origin.

Define, for each $y \in F$, the projection set

$$\pi_y(E) = \{x \cdot y : x \in E\}.$$

Suppose that for some $0 < \alpha \leq 1$,

$$\max \left\{ \frac{\min\{\gamma_F, s_E\}}{\alpha}, \frac{s_E + s_F - l_F + 1 - \alpha}{d} \right\} > 1. \quad (1.4)$$

Then

$$\dim_{\mathcal{H}}(\pi_y(E)) \geq \alpha$$

for μ_F -every $y \in F$.

If (1.4) holds with $\alpha = 0$, then

$$\mathcal{L}^1(\pi_y(E)) > 0$$

for μ_F -every $y \in F$.

Remark 1.0.4. Observe that the conditions of Theorem 1.0.3 always hold with $\gamma_F = 0$ and $l_F = 1$ since every tube T_δ can be decomposed into $\approx \delta^{-1}$ balls of radius δ .

Corollary 1.0.5. Let $A \subset \mathbb{R}$ and let μ_A be a Frostman measure on A . Then the following hold:

- Suppose that the Hausdorff dimension of A , denoted by $\dim_{\mathcal{H}}(A)$, is greater than $\frac{1}{2} + \epsilon$ for some $0 < \epsilon \leq \frac{1}{2}$. Then for $\mu_A \times \mu_A \times \cdots \times \mu_A$ -every $(a_1, a_2, \dots, a_d) \in A \times A \times \cdots \times A$,

$$\dim_{\mathcal{H}}(a_1 A + a_2 A + \cdots + a_d A) \geq \min \left\{ 1, \frac{1}{2} + \epsilon(2d - 1) \right\}. \quad (1.5)$$

- Suppose that the Hausdorff dimension of A is greater than

$$\frac{1}{2} + \frac{1}{2(2d - 1)}$$

for some $d \geq 2$. Then for $\mu_A \times \mu_A \times \cdots \times \mu_A$ -every $(a_1, a_2, \dots, a_d) \in A \times A \times \cdots \times A$,

$$\mathcal{L}^1(a_1 A + a_2 A + \cdots + a_d A) > 0. \quad (1.6)$$

- Suppose that $F \subset \mathbb{R}^d$, $d \geq 2$, is star-like in the sense that the intersection of F with every tube of width δ containing the origin is contained in a fixed number of balls of radius δ . Assume that

$$\dim_{\mathcal{H}}(A) + \frac{\dim_{\mathcal{H}}(F)}{d} > 1.$$

Then for μ_F -every $x \in F$,

$$\mathcal{L}^1(x_1A + x_2A + \cdots + x_dA) > 0. \quad (1.7)$$

In particular, if $\dim_{\mathcal{H}}(A) = \frac{1}{2} + \epsilon$, for some $\epsilon > 0$, and $\dim_{\mathcal{H}}(F) > \frac{d}{2} - \epsilon$, then (1.7) holds for μ_F -every $x \in F$.

- Suppose that $F \subset \mathbb{R}^d$, $d \geq 2$, possesses a Borel measure μ_F such that (1.2) holds with $\gamma_F > 1$. Suppose that $\dim_{\mathcal{H}}(A) > \frac{1}{2}$. Then for μ_F -every $x \in F$,

$$\mathcal{L}^1(x_1A + x_2A + \cdots + x_dA) > 0.$$

1.1 Applications to the finite point configurations

Recall that the celebrated Falconer distance conjecture ([7]) says that if the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, is $> \frac{d}{2}$, then the Lebesgue measure of the set of distances $\{|x - y| : x, y \in E\}$ is positive. The best known result to date, due to Wolff ([21]) in two dimension and Erdogan ([5]) in higher dimensions says that the Lebesgue measure of the distance set is indeed positive if the Hausdorff dimension of E is greater than $\frac{d}{2} + \frac{1}{3}$.

Corollary 1.1.1. *Let*

$$E \subset S^{d-1} = \left\{ x \in \mathbb{R}^d : \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2} = 1 \right\}$$

of Hausdorff dimension $> \frac{d}{2}$. Let μ_E be a Frostman measure on E . Then

$$\mathcal{L}^1(\{|x - y| : x \in E\}) > 0$$

for μ -every $y \in E$.

Before stating our next result, we need the following definition. Let $T_k(E)$, $1 \leq k \leq d$, denote the $(k+1)$ -fold Cartesian product of E with the equivalence relation where $(x^1, \dots, x^{k+1}) \sim (y^1, \dots, y^{k+1})$ if there exists a translation τ and an orthogonal transformation O such that $y^j = O(x^j) + \tau$.

In analogy with the Falconer distance conjecture, it is reasonable to ask how large the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, needs to be to ensure that the $\binom{k+1}{2}$ -dimensional Lebesgue measure of $T_k(E)$ is positive.

Theorem 1.1.2. *Let $E \subset S_t = \{x \in \mathbb{R}^d : |x| = t\}$ for some $t > 0$. Suppose that $\dim_{\mathcal{H}}(E) > \frac{d+k-1}{2}$. Then*

$$\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0.$$

Using a pigeon-holing argument, we obtain the following result for finite point configurations in \mathbb{R}^d .

Corollary 1.1.3. *Let $E \subset \mathbb{R}^d$ of Hausdorff dimension $> \frac{d+k+1}{2}$. Then*

$$\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0.$$

Remark 1.1.4. We do not know to what extent our results are sharp beyond the following observations. If the Hausdorff dimension of E is less than $\frac{d}{2}$, the classical example due to Falconer ([7]) shows that the set of distances may have measure 0, so, in particular, $\mathcal{L}^{\binom{k+1}{2}}(T_k(E))$ may be 0 for $k > 1$. See also [6] and [16] for the description of the background material and [15] and [13] for related counter-example construction. In two dimensions, one can generalize Falconer's example to show that if the Hausdorff dimension of E is less than $\frac{3}{2}$, then the three dimensional Lebesgue measure of $T_2(E)$ may be zero. In higher dimension, construction of examples of this type is fraught with serious number theoretic difficulties. We hope to address this issue in a systematic way in the sequel.

2 Proofs of main results

2.1 Proof of the projection results (Theorem 1.0.3)

Define the measure ν_y on $\pi_y(E)$ by the relation

$$\int g(s) d\nu_y(s) = \int g(x \cdot y) d\mu_E(x).$$

It follows that

$$\int \int |\widehat{\nu}_y(t)|^2 dt d\mu_F(y) = \int \int |\widehat{\mu}_E(ty)|^2 d\mu_F(y) dt.$$

We have

$$\begin{aligned} \int |\widehat{\mu}_E(ty)|^2 d\mu_F(y) &= \int \int \widehat{\mu}_E(t(u-v)) d\mu_E(u) d\mu_E(v) \\ &= \int \int_{|u-v| \leq t^{-1}} \widehat{\mu}_E(t(u-v)) d\mu_E(u) d\mu_E(v) \\ &+ \int \int_{|u-v| > t^{-1}} \widehat{\mu}_E(t(u-v)) d\mu_E(u) d\mu_E(v) = I + II. \end{aligned}$$

Since μ_E is a Frostman measure,

$$|I| \lesssim t^{-s_E}.$$

By (1.2),

$$|II| \lesssim t^{-\gamma_F} \int \int |u - v|^{-\gamma_F} d\mu(u) d\mu(v) \lesssim t^{-\gamma_F}$$

as long as $\gamma_F \leq s_E$. If $\gamma_F > s_E$, then for any $\epsilon > 0$,

$$|II| \lesssim t^{-s_E+\epsilon} \int \int |u - v|^{-s_E+\epsilon} d\mu(u) d\mu(v) \lesssim t^{-s_E},$$

and we conclude that

$$\int |\widehat{\mu}_E(ty)|^2 d\mu_F(y) \lesssim t^{-\min\{s_E, \gamma_F\}}. \quad (2.1)$$

It follows that

$$\int \int |\widehat{\nu}_y(t)|^2 t^{-1+\alpha} d\mu_F(y) dt < \infty$$

if $\min\{s_E, \gamma_F\} > \alpha$.

We now argue via the uncertainty principle. We may assume, by scaling and pigeonholing, that $F \subset \{x \in \mathbb{R}^d : 1 \leq |x| \leq 2\}$. Let ϕ be a smooth cut-off function supported in the ball of radius 3 and identically equal to 1 in the ball of radius 2. It follows that

$$\begin{aligned} & \int |\widehat{\nu}_y(t)|^2 t^{-1+\alpha} dt \\ &= \int \int |\widehat{\mu}_E(ty)|^2 t^{-1+\alpha} dt d\mu_F(y) = \int \int |\widehat{\mu}_E * \widehat{\phi}(ty)|^2 d\mu_F(y) t^{-1+\alpha} dt \\ &\lesssim \int \int \int |\widehat{\mu}_E(\xi)|^2 |\widehat{\phi}(ty - \xi)| d\mu_F(y) t^{-1+\alpha} dt d\xi \\ &\lesssim \sum_m 2^{-mn} \int |\widehat{\mu}_E(\xi)|^2 \mu_F \times \mathcal{L}^1\{(y, t) : |ty - \xi| \leq 2^m\} |\xi|^{-1+\alpha} d\xi \\ &\lesssim \sum_m 2^{-mn} \cdot 2^m \int |\widehat{\mu}_E(\xi)|^2 \mu_F \left(T_{2^m|\xi|^{-1}}(\xi) \right) |\xi|^{-1+\alpha} d\xi, \end{aligned}$$

where $T_\delta(\xi)$ is the tube of width δ and length 10 emanating from the origin in the direction of $\frac{\xi}{|\xi|}$. By assumption, this expression is

$$\lesssim \sum_m 2^{-mn} \cdot 2^m \cdot 2^{m(s_F - l_f)} \int |\widehat{\mu}_E(\xi)|^2 |\xi|^{-s_F + l_F - 1 + \alpha} d\xi \lesssim 1$$

if

$$s_F - l_F + 1 - \alpha > d - s_E$$

and n is chosen to be sufficiently large. Combining this with (2.1) we obtain the conclusion of Theorem 1.0.3.

2.2 Proof of applications to sums and products (Corollary 1.0.5)

Let $A \subset \mathbb{R}$ have dimension greater than $s_A := \frac{1}{2} + \frac{1}{2(2d-1)}$. Note that we can find a probability measure, μ_A , supported on A satisfying

$$\mu_A(B(x, r)) \leq Cr^{s_A}, \quad x \in \mathbb{R}, r > 0, \quad (2.2)$$

Let $E = A \times A \times \cdots \times A$. Define $\mu_E = \mu_A \times \mu_A \times \cdots \times \mu_A$.

Lemma 2.2.1. *With the notation above,*

$$\mu_E(T_\delta) \lesssim \delta^{(d-1)s_A}, \quad (2.3)$$

where $\dim_{\mathcal{H}}(A) = s_A$.

Proof. Let $l_\xi = \{s\xi : s \in \mathbb{R}\}$ and assume without loss of generality that ξ_1 is the largest coordinate of ξ in absolute value. In particular, $\xi_1 \neq 0$. Define the function

$$\Psi : A \rightarrow (A \times A \times \cdots \times A) \cap l_\xi$$

by the relation

$$\Psi(a) = \left(a, a \frac{\xi_2}{\xi_1}, \dots, a \frac{\xi_d}{\xi_1} \right).$$

Note that

$$\begin{aligned} \mu(T_\delta(\xi)) &= \mu_A \times \cdots \times \mu_A(T_\delta(\xi)) \\ &\leq \int_{-10}^{10} \mu_A \times \cdots \times \mu_A(B(\Psi(x_1), \delta)) d\mu_A(x_1) \\ &\lesssim \int_{-10}^{10} \delta^{(d-1)s_A} d\mu_A(x_1) \lesssim \delta^{(d-1)s_A}. \end{aligned}$$

□

It follows that if $E = F = A \times A \times \cdots \times A$, then the assumptions of Theorem 1.0.3 are satisfied with $s_E = s_F = s_A$, $\gamma_F = 0$ and $l_F = \frac{s_E}{d}$. The conclusion of the first part of Corollary 1.0.5 follows in view of Theorem 1.0.3.

The second conclusion of Corollary 1.0.5 follows, in view of Theorem 1.0.3, if we observe that if F is star-like, then

$$\mu_F(T_\delta) \lesssim \delta^{-s_F}.$$

The third conclusion of Corollary 1.0.5 follows from Theorem 1.0.3 since we may always take $l_F = 1$. This holds since every tube δ is contained in a union of δ^{-1} balls of radius δ .

2.3 Proof of the spherical configuration result (Theorem 1.1.2)

Let $y = (y^1, y^2, \dots, y^k)$, $y^j \in E$ and define

$$\pi_y(E) = \{x \cdot y^1, \dots, x \cdot y^k : x \in E\}.$$

Define a measure on $\pi_y(E)$ by the relation

$$\int g(s) d\nu_y(s) = \int g(x \cdot y^1, \dots, x \cdot y^k) d\mu(x),$$

where $s = (s_1, \dots, s_k)$ and μ is a Frostman measure on E . It follows that

$$\widehat{\nu}_y(t) = \widehat{\mu}(t \cdot y),$$

where $t = (t_1, \dots, t_k)$ and

$$t \cdot y = t_1 y^1 + t_2 y^2 + \dots + t_k y^k.$$

It follows that

$$\int \int |\widehat{\nu}_y(t)|^2 dt d\mu^*(y) = \int \int |\widehat{\mu}(t \cdot y)|^2 dt d\mu^*(y),$$

where

$$d\mu^*(y) = d\mu(y^1) d\mu(y^2) \dots d\mu(y^k).$$

Arguing as above, this quantity is bounded by

$$\int \int \int |\widehat{\mu}(\xi)|^2 |\widehat{\phi}(t \cdot y - \xi)| dt d\mu^* d\xi.$$

It is not difficult to see that

$$\int \int |\widehat{\phi}(t \cdot y - \xi)| dt d\mu^* \lesssim |\xi|^{-s+k-1}$$

since once we fix a linearly independent collection y^1, y^2, \dots, y^{k-1} , y^k is contained in the intersection of E with a k -dimensional plane. Since E is also a subset of a sphere, the claim follows. If y^1, \dots, y^{k-1} are not linearly independent, the estimate still holds, but the easiest way to proceed is to observe that since the Hausdorff dimension of E is greater than k by assumption, there exist $k-1$ disjoint subsets E_1, E_2, \dots, E_{k-1} of E and a constant $c > 0$ such that $\mu(E_j) \geq c$ and any collection $y^1, \dots, y^{k-1}, y^j \in E_j$ is linearly independent. This establishes our claim with μ^* replaced by the product measure restricted to E_j s, which results in the same conclusion.

Plugging this in we get

$$\int |\widehat{\mu}(\xi)|^2 |\xi|^{-s+k-1} d\xi < \infty$$

if

$$s > \frac{d}{2} + \frac{k-1}{2}.$$

This implies that for μ^k almost every k -tuple $y = (y^1, y^2, \dots, y^k) \in E^k$, ν_y is absolutely continuous w.r.t \mathcal{L}^k and hence its support, $\pi_y(E)$, is of positive \mathcal{L}^k measure.

Let

$$\mathcal{E}_k = \{y = (y^1, \dots, y^k) \in E^k : \mathcal{L}^k(\pi_y(E)) > 0\}.$$

We just proved that $\mu^k(\mathcal{E}_k) = \mu^k(E^k) > 0$.

Consider the set

$$P_{k-1} = \{(y^1, \dots, y^{k-1}) \in E^{k-1} : \mu(\{x : (y^1, \dots, y^{k-1}, x) \in \mathcal{E}_k\}) = \mu(E)\}.$$

By the discussion above, and Fubini, $\mu^{k-1}(P_{k-1}) = \mu(E)^{k-1} > 0$.

For each $y = (y^1, \dots, y^{k-1}) \in P_{k-1}$, let $F_y = \{x \in E : (y^1, \dots, y^{k-1}, x) \in \mathcal{E}_k\}$, and define

$$\pi_y(F_y) = \{(x \cdot y^1, \dots, x \cdot y^{k-1}) : x \in F_y\}.$$

As above we construct the measure ν_y supported on $\pi_y(F_y)$, and we have

$$\widehat{\nu_y}(t) = \widehat{\mu \chi_{F_y}}(t \cdot y) = \widehat{\mu}(t \cdot y),$$

since $\mu(E \setminus F_y) = 0$. By the argument above, we conclude that $\pi_y(F_y)$ is of positive \mathcal{L}^{k-1} measure. Therefore, by Fubini, for μ^{k-1} a.e. $(y^1, \dots, y^{k-1}) \in E^{k-1}$,

$$\mathcal{L}^{k+k-1}\{y^1 \cdot y, \dots, y^{k-1} \cdot y, y^1 \cdot x, \dots, y^{k-1} \cdot x, y \cdot x : x, y \in E\} > 0.$$

The result now follows from this induction step.

3 Proof of the Euclidean configuration result (1.1.3)

We shall make use of the following intersection result. See Theorem 13.11 in [16].

Theorem 3.0.1. *Let $a, b > 0$, $a + b > d$, and $b > \frac{d+1}{2}$. If A, B are Borel subsets in \mathbb{R}^d with $\mathcal{H}^a(A) > 0$ and $\mathcal{H}^b(B) > 0$, then for almost all $g \in O(d)$,*

$$\mathcal{L}^d\{z \in \mathbb{R}^d : \dim_{\mathcal{H}}(A \cap (\tau_z \circ g)B) \geq a + b - d\} > 0,$$

where τ_z denotes the translation by z .

In the special case when B is the unit sphere in dimension 4 or higher, we get the following corollary.

Corollary 3.0.2. *Let $E \subset \mathbb{R}^d$, $d \geq 4$, of Hausdorff dimension $s > 1$. Then*

$$\mathcal{L}^d \{z \in \mathbb{R}^d : \dim_{\mathcal{H}} (E \cap (S^{d-1} + z)) \geq s - 1\} > 0.$$

It follows from the corollary that if $E \subset \mathbb{R}^d$, $d \geq 4$, is of Hausdorff dimension $> \frac{d+k+1}{2}$, there exists $z \in \mathbb{R}^d$, such that the Hausdorff dimension of $E \cap (z + S^{d-1})$ is $> \frac{d+k-1}{2}$. Now observe that if $x, y \in z + S^{d-1}$, means that $x = x' + z$, $y = y' + z$, where $x', y' \in S^{d-1}$. It follows that

$$|x - y|^2 = |x' - y'|^2 = 2 - 2x' \cdot y'.$$

In other words, the problem of simplexes determined by elements of $E \cap (z + S^{d-1})$ reduces to Theorem 1.1.2 and thus Theorem 1.1.3 is proved.

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